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A BRST analysis of W-symmetries

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We perform a classical BRST analysis of the symmetries corresponding to a generic w_N-algebra. An essential feature of our method is that we write the w_N-algebra in a special basis such that the algebra manifestly has a "nested" set of subalgebras $v_N^N \subset v_N^{N-1} \subset \ldots \subset v_N^2 \equiv w_N$ where the subalgebra v_N^i $(i = 2, \ldots, N)$ consists of generators of spin $s = \{i, i+1, \ldots, N\}$, respectively. In the new basis the BRST charge can be written as a "nested" sum of $N-1$ nilpotent BRST charges. In view of potential applications to (critical and/or non-critical) W-string theories we discuss the quantum extension of our results. In particular, we present the quantum BRST operator for the W_4-algebra in the new basis. For both critical and non-critical W-strings we apply our results to discuss the relation with minimal models.

1. Introduction

In recent years it has turned out that in order to describe string theory it is convenient to use the BRST formalism [1]. For instance, via a BRST analysis one can derive the critical dimension and calculate the spectrum of the theory. For critical strings this was first done in ref. [2]. More recently, the spectrum of non-critical strings has been calculated using this formalism [3]. The BRST approach also plays a crucial role in the construction of a string field theory [4].

The starting point in the BRST approach is the introduction of a set of canonical variables (the "string coordinates") which satisfy a standard Poisson bracket. In string theory the relevant variables are given by a set of holomorphic variables and a set of anti-holomorphic variables. We restrict the BRST analysis to the holomorphic sector since the two sectors require a similar treatment. The two-dimensional conformal symmetries of string theory are encoded in a set of first-class constraints on the string coordinates whose Poisson brackets ar given by the Virasoro algebra. Given this Virasoro algebra one can construct a nilpotent BRST charge by extending the phase space with a set of anticommuting ghost variables. At the classical level, this BRST charge can be used to define the

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physical variables of the theory. In a canonical quantization the Poisson brackets get replaced by so-called Operator Product Expansions (OPEs) where the operators act in a Hilbert space. At the same time the BRST charge gets replaced by a nilpotent BRST operator. The physical states in the Hilbert space are defined as the cohomology classes of this BRST operator. The BRST operator thus provides a convenient way to calculate the spectrum of the theory.

Due to normal ordering problems it is not guaranteed *a priori* that a nilpotent BRST operator can be constructed. If this is not the case one cannot define the physical states and the theory is said to be anomalous. In most cases the BRST operator can be made nilpotent provided that certain conditions hold. For instance, in the case of the bosonic critical string requiring nilpotency of the BRST operator leads to the condition that the number of string coordinates is 26, i.e. the bosonic string moves in a 26-dimensional spacetime [2].

Within the BRST formalism it is rather natural to extend the Virasoro constraints with a set of additional first-class constraints and investigate whether this extended set still leads to a sensible spectrum thus providing the basis for the construction of new string theories [5]. The complete set of first-class constraints must form a closed Poisson-bracket algebra which is an extension of the classical Virasoro algebra *. Most of the recent research has focussed on algebras where the new generators carry a spin which is higher than the spin of the Virasoro generators. Such algebras are denoted as extended conformal algebras or, briefly, "w_N-algebras" where N indicates the highest spin of the generators involved (usually one uses a convention in which the Virasoro generators carry a spin equal to two). The simplest example, which has been mostly studied, is the w_3-algebra which involves the Virasoro generators and a generator of spin three [6]. The w_3-algebra is quadratically nonlinear, i.e., Poisson brackets of the constraints lead to polynomials of the constraints which are at most quadratic. The BRST charge of the w_3-algebra was first constructed in ref. [7] while the BRST charge for general quadratically nonlinear algebras was obtained in ref. [8].

In view of potential applications to W-string theories it is necessary to quantize the w_N-symmetries via the BRST formalism and to perform a spectrum analysis. One noteworthy feature that has emerged from this quantization is that although classically the first-class constraints always form a closed Poisson-bracket algebra, the corresponding quantum operators do not necessarily form a closed quantum algebra in the full Hilbert space, even after including possible renormalizations of

* A few clarifying remarks concerning the terminology "classical" algebras are in order here. In general, by a classical algebra is meant a Poisson-bracket algebra. In this sense there exists a classical Virasoro algebra with a so-called central extension. However, in this paper we will always reserve the term "classical" algebra for the special case where this central extension is zero. For the realization of the Virasoro algebra in terms of free fields this means that we do not consider background charges at the classical level. Similarly, by a classical w_N-algebra (see below) we mean a Poisson-bracket algebra whose free field realization contains only *single* derivatives of the fields.

the generators and allowing for quantum deformations of the classical algebra *. Indeed, they do not have to form a closed quantum algebra. All one needs in the BRST approach at the quantum level is the existence of a nilpotent BRST operator. So we have the following picture:

$$\begin{aligned} &\text{classical} &\rightarrow\quad &\text{closed Poisson-bracket algebra,} \\ &\text{quantum} &\rightarrow\quad &\text{nilpotent BRST operator.} \end{aligned} \tag{1}$$

A recent example of a nilpotent BRST operator without a corresponding quantum algebra was given in ref. [9]. In the present work we will encounter more examples. Once a nilpotent BRST operator has been constructed, its cohomology, and hence the spectrum of the theory, can be computed. The quantum constraints, which by construction are BRST-trivial, then close within the space of cohomology classes of the BRST operator.

It is the purpose of this paper to give a systematic BRST analysis of general w_N-symmetries both at the classical as well as at the quantum level. So far, explicit results are known and well understood only in the case of the w_3-algebra. In refs. [10,11], an expression has been presented for the BRST operator of the w_4-algebra. However, the complexity of this expression makes it rather hard to deal with in practice. Recently it has been pointed out that in case of the w_3-algebra the BRST analysis can be simplified by making an appropriate redefinition of the canonical variables [12]. After the redefinition the BRST charge can be written as the sum of two charges that are separately nilpotent. It is expected that this will lead to simplifications in the analysis of the spectrum in the quantum case. In ref. [13] the redefinition of the canonical variables was translated into a corresponding redefinition of the generators and it was indicated how a similar simplification could be made for the generic w_N-algebra. The additional structure which arises after the redefinitions makes it possible to obtain a relatively simple structured expression for the BRST operator for W_4 (see sect. 5), and in principle also for W_N.

The general picture that arises and which is confirmed by the present work is as follows. Usually the w_N-algebra is realized in terms of $N-1$ free scalars fields and given in a special basis which is related to making a so-called Miura transformation. We will call this special basis the "Miura basis". In this Miura basis the BRST charge of the w_N-algebra is a rather complicated expression which for growing N contains terms of increasingly high order in the ghost fields. For instance, the BRST charge of the w_3-algebra is at most trilinear in the ghosts but the BRST charge of the w_4-algebra (see sect. 4) contains already terms of seventh

* To be more precise, the existence of a quantum algebra depends on the basis one is using for the classical algebra. Using the standard, so-called Miura (see below), basis of the w_N-algebra, there exists a corresponding quantum algebra which we denote by W_N. This is however not the case if we use our new, realization-dependent, basis of the w_N-algebra (see below).

order in the ghosts. In the next section we will show how the generators of the w_N-algebra can be redefined such that the w_N-algebra contains a "nested" set of subalgebras

$$v_N^N \subset v_N^{N-1} \subset \ldots \subset v_N^2 \equiv w_N, \tag{2}$$

where the subalgebra v_N^i $(i = 2, \ldots, N)$ consists of $N-i+1$ generators $\{w_N^i, w_N^{i+1}, \ldots, w_N^N\}$ of spin $s = \{i, i+1, \ldots, N\}$, respectively. The generators are realized by $N-1$ free (holomorphic) scalar fields ϕ_n, $n = 1, \ldots, N-1$, such that the generator w_N^N, of highest spin, only depends on the single scalar ϕ_{N-1}, the generator w_N^{N-1}, of nest to highest spin, only depends on the two scalars ϕ_{N-1}, ϕ_{N-2}, etc. Finally, the Virasoro generator w_N^2 is the only generator that depends on all scalars $\phi_1, \ldots, \phi_{N-1}$. This particular dependence of the generators on the scalars automatically leads to the nested subalgebra structured indicated in (2). For instance, since the highest spin generator w_N^N only depends on ϕ_{N-1}, and all other generators contain other scalars as well, the Poisson-bracket algebra of w_N^N must close on itself thus leading to the subalgebra v_N^N etc.

An immediate consequence of the new basis is that the scalar ϕ_1 which only occurs in the Virasoro generator can be replaced there by a term $\partial X^\mu \partial X_\mu$ containing an arbitrary number of scalars X^μ without upsetting the closure of the algebra since this term commutes with all the other generators. This leads to a multi-scalar realization of the w_N-algebra. Such multi-scalar realizations were first considered in ref. [14]. The above structure is summarized schematically in table 1.

In order to construct the BRST charge of the complete w_N-algebra one can now first consider the smallest subalgebra v_N^N generated by w_N^N. Its corresponding BRST charge we denote by Q_N^N. One then considers the next subalgebra v_N^{N-1} generated by $\{w_N^N, w_N^{N-1}\}$ which has its own BRST charge Q_N^{N-1}. Since $v_N^N \subset v_N^{N-1}$ we have that $Q_N^N \subset Q_N^{N-1}$. By this we mean that if one sets the ghost variables corresponding to the spin-$(N-1)$ symmetries equal to zero the expression for Q_N^{N-1} equals that of Q_N^N. In general, this does not imply that the BRST charge

TABLE 1

This table shows the generic structure of the w_N-algebra in the new basis discussed in sect. 2. The left column indicates the generators $\{w_N^2, \ldots, w_N^N\}$ of the algebra. The other columns indicate the dependence of the generators on the scalars $\{X^\mu, \phi_2, \ldots, \phi_{N-1}\}$

Generator	Dependent on				
w_N^2	X^μ	ϕ_2	ϕ_3	$\cdots$	ϕ_{N-1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	
w_N^{N-2}	ϕ_{N-3}	ϕ_{N-2}	ϕ_{N-1}		
w_N^{N-1}	ϕ_{N-2}	ϕ_{N-1}			
w_N^N	ϕ_{N-1}				

Q_N^{N-1} can be written as $Q_N^{N-1} = Q_N^N +$ "rest" such that the "rest" terms are separately nilpotent. The fact that this does happen for the w_3-algebra is an exception (see below). We thus arrive at the following "nested" structure of the BRST charge Q_N of the w_N-algebra:

$$Q_N^N \subset Q_N^{N-1} \subset Q_N^{N-2} \subset \ldots \subset Q_N^3 \subset Q_N^2 \equiv Q_N. \tag{3}$$

Here the inclusion symbols indicate how the different (nilpotent) BRST charges can be obtained from each other by setting certain ghost variables equal to zero. A nice feature of this structure is that one can investigate systematically the BRST charges of the different nested subalgebras and thus iteratively construct the BRST charge of the complete w_N-algebra. In this paper we will present results for the subalgebras v_N^N and v_N^{N-1} for *any N*.

Note that the generator w_N^2 always satisfies by itself the Virasoro algebra. Therefore there is, besides the nested structure (3), also a nilpotent BRST charge corresponding to this Virasoro subalgebra. This BRST charge is in fact given by $Q_N^2 - Q_N^3$. This is the reason that for $N = 3$ the nested structure (3) is given by

$$Q_3^2 = Q_0 + Q_1, \tag{4}$$

where $Q_0 = Q_3^2 - Q_3^3$ and $Q_1 = Q_3^3$ are two anticommuting nilpotent BRST charges [12,13].

It is to be expected that the nested structure (3) of the BRST charges survives quantization *. The examples given in this paper provide arguments in favour of this conjecture. In that case the nested structure discussed in this paper should be useful in the construction of the spectrum of the W_N-string.

In refs. [15,16], a relationship was suggested between the spectra of W_N-strings and Virasoro minimal models. In the case of the W_3-string this relation has been made more explicit in refs. [12,17–20]. In particular, it was shown that the W_3-string can be viewed as an ordinary $c = 26$ string, where the matter sector includes a $c = \frac{1}{2}$ Ising model. From the point of view of the nested structure (3), it is easy to see how the $c = \frac{1}{2}$ Ising model enters into the game by observing the following numerology. Since the v_3^3 subalgebra has its own nilpotent BRST operator $\mathbf{Q}_3^3$, one can separately construct its cohomology. The BRST operator $\mathbf{Q}_3^3$ is realized by a single free scalar ϕ_2 and the ghosts of the spin-three symmetries. It turns out that the total central charge c_3^3 of these fields equals $\frac{1}{2}$ which is precisely that of the Ising model. In this paper we will apply a similar numerology to the nested structure of a generic w_N-algebra. Our results suggest a very general relationship between the spectra of W_N-strings and W minimal models. A similar relationship is suggested between the so-called non-critical W_N-strings and W

* To distinguish between classical and quantum expressions, we will write the quantum expressions with boldface.

minimal models, thereby extending a conjecture made in ref. [13]. It will be interesting to see whether the conjectures will be confirmed by explicit calculations of the spectra of (critical and/or non-critical) W_N-strings. We hope that the nested structure discussed in this paper will considerably facilitate this task.

The organization of this paper is as follows. In sect. 2 we show how the redefinition of the w_N-algebra discussed above can be carried out for arbitrary N. In sect. 3 we present general results for any N for the first two subalgebras v_N^N and v_N^{N-1}. In sect. 4 we discuss the special cases $N=3, 4, 5$. The discussion of sects. 2, 3 and 4 is always at the classical level. In sect. 5 we extend some of our results to the quantum case. For instance, for $N=4$ we give the quantum BRST operator corresponding to the w_4-algebra. Finally, in sect. 6 we discuss the relations with (Virasoro) minimal models for both critical W_N-strings and the non-critical W_N-strings of ref. [9].

2. A new basis for the w_N-algebra

In this section we will introduce the new basis for the w_N-algebra, starting from realizations of the w_N-algebra obtained from the Miura transformation [21]. The basic result of this section is given by formulae (27), (34) where we give a closed expression for *all* generators of the classical w_N-algebra in the new basis described in the introduction.

The Miura transformation generates realizations of w_N in terms of $N-1$ scalar fields ϕ_n, $n=1,\dots,N-1$. This construction is iterative in the sense that the generators of the w_{N+1}-algebra can be expressed in terms of those of the w_N-algebra and one additional scalar field ϕ_N [15,22]. We will denote the generators of w_N in the Miura basis by M_N^l, where l is the spin, $2\leqslant l\leqslant N$. The iterative structure induced by the Miura transformation reads

$$M_{N+1}^l=\sum_{k=0}^{l} a_{l,k}{}^{N+1}(B_N)^{l-k}M_N^k, \tag{5}$$

where it is assumed that $M_N^k=0$ for $k>N$. B_n represents the scalar field ϕ_n:

$$B_n=\frac{i}{\sqrt{2n(n+1)}}\partial\phi_n, \tag{6}$$

and the coefficients a in (5) are given by

$$k\leqslant l \qquad a_{l,k}{}^{N+1}=(-1)^{l-k}\frac{[N-l+1-N(l-k)](N-k)!}{(N-l+1)!(l-k)!}, \tag{7}$$

$$l<k\leqslant N \qquad a_{l,k}{}^{N+1}=0. \tag{8}$$

TABLE 2
Generators of w_N in the Miura basis for some low values of N

N	Generators of w_N
2	$M_2^2=-(B_1)^2$
3	$M_3^2=M_2^2-3(B_2)^2$ $M_3^3=2[B_2M_2^2+(B_2)^3]$
4	$M_4^2=M_3^2-6(B_3)^2$ $M_4^3=M_3^3+2B_3M_3^2+8(B_3)^3$ $M_4^4=3[B_3M_3^3-(B_3)^2M_3^2-(B_3)^4]$
5	$M_5^2=M_4^2-10(B_4)^2$ $M_5^3=M_4^3+2B_4M_4^2+20(B_4)^3$ $M_5^4=M_4^4+3B_4M_4^3-7(B_4)^2M_4^2-15(B_4)^4$ $M_5^5=4[B_4M_4^4-(B_4)^2M_4^3+(B_4)^3M_4^2+(B_4)^5]$

Eq. (5) generates realizations of the classical w_N-algebra starting from $M_0^0=1$, $M_1^1=0$. Note that in particular (5) then implies that

$$M_N^0=1, \tag{9}$$

$$M_N^1=0, \tag{10}$$

$$M_N^2=-\sum_{n=1}^{N-1}\tfrac{1}{2}n(n+1)(B_n)^2. \tag{11}$$

The standard form of the energy–momentum tensor is then obtained as $T=-2M_N^2$. To illustrate the Miura basis we give explicit results for the generators of w_N, $N=2$, 3, 4, 5 in table 2.

The generators M_N^l at fixed N form a closed Poisson-bracket algebra. Clearly, this is then also the case for any linear combination of the M_N^l. The redefinition we will now discuss uses the iterative structure (5) to simplify the generators by making appropriate linear combinations. The aim is to construct a set of generators such that the highest spin depends on only one scalar, B_{N-1}, the next highest spin on two scalars, etc.

As an example, let us perform this redefinition explicitly for the first nontrivial case, $N=4$. We start with the highest spin generator, M_4^4. As we see in table 2, it depends on M_3^3 and M_3^2. However, these can be expressed in terms of M_4^3 and M_4^2 by inverting the relations given in table 2:

$$M_3^2=M_4^2+6(B_3)^2, \tag{12}$$

$$M_3^3=M_4^3-2B_3M_4^2-20(B_3)^3. \tag{13}$$

This we substitute in the expression for M_4^4 to obtain

$$M_4^4 = 3\left[B_3 M_4^3 - 3(B_3)^2 M_4^2 - 27(B_3)^4\right]. \tag{14}$$

The new spin-four generator w_4^4 is then defined as the linear combination

$$\begin{aligned} w_4^4 &= M_4^4 - 3\left[B_3 M_4^3 - 3(B_3)^2 M_4^2\right] \\ &= -81(B_3)^4. \end{aligned} \tag{15}$$

Now let us define w_4^3. We get from table 2

$$M_4^3 = M_3^3 + 2B_3 M_3^2 + 8(B_3)^3. \tag{16}$$

To express M_3^3 in terms of M_4^l, $l < 3$, we must first make use of the $N = 3$ entries in table 2. These allow us to express M_3^3 in terms of M_3^2:

$$M_3^3 = 2B_2 M_3^2 + 8(B_2)^3. \tag{17}$$

This, and (12), is then substituted in (16). The result is

$$M_4^3 = 2(B_2 + B_3)\left[M_4^2 + 6(B_3)^2\right] + 8(B_2)^3 + 8(B_3)^3. \tag{18}$$

For M_4^3 we now make a redefinition which gets rid of M_4^2. The resulting spin-three generator w_4^3 is

$$w_4^3 = 8(B_2)^3 + 12B_2(B_3)^2 + 20(B_3)^3. \tag{19}$$

After employing a similar procedure for the spin-two generator we find that there is no redefinition to be made. The result is

$$w_4^2 = M_4^2 = -(B_1)^2 - 3(B_2)^2 - 6(B_3)^2. \tag{20}$$

The algorithm relies on the use of the inverse of (5). To complete the redefinition for w_4 required the inverse of ${a_{l,k}}^{N+1}$ for all $N < 4$.

Let us now consider the above algorithm for general N. We start with the highest spin of the w_{N+1}-algebra. From (5) and (7) we obtain for this generator

$$M_{N+1}^{N+1} = \sum_{l=0}^{N} (-1)^{N-l} N(B_N)^{N+1-l} M_N^l. \tag{21}$$

Now, (5) expresses the generators of w_{N+1} in terms of those of w_N, but, as in (12), (13), we can use (5) in the opposite direction to express the M_N^l, $l = 0, \ldots, N$, in terms of M^k_{N+1}, $k=0,\dots,N$. As we saw in the w_4-example above, this requires the inverse of the $(N+1)\times(N+1)$ lower-triangular matrix $a_{l,k}{}^{N+1}$, l, $k=0,\dots,N$. The inverse takes on the following form:

$$k\leqslant l \qquad f_{l,k}{}^{N+1}=\sum_{m=0}^{l-k}\binom{N-k-m}{l-k-m}(-N)^m$$

$$=\binom{N-k}{l-k}{}_2F_1(1,\ -l+k;\ -N+k;\ -N), \tag{22}$$

$$l<k\leqslant N \qquad f_{l,k}{}^{N+1}=0. \tag{23}$$

The inverse of (5) then becomes

$$M^l_N=\sum_{k=0}^{N} f_{l,k}{}^{N+1}(B_N)^{l-k}M^k_{N+1}. \tag{24}$$

This we can substitute in (21), to obtain

$$M^{N+1}_{N+1}=(-1)^N N\sum_{k=0}^{N}(B_N)^{N+1-k}M^k_{N+1}\sum_{l=0}^{N}(-1)^l f_{l,k}{}^{N+1}$$

$$=-\sum_{k=0}^{N}(-NB_N)^{N+1-k}M^k_{N+1}. \tag{25}$$

Here we have used the following result for the coefficients f:

$$\sum_{l=0}^{N}(-1)^l f_{l,k}{}^{N+1}=(-1)^k N^{N-k}. \tag{26}$$

Now we can redefine the highest spin (we will denote the spin-l generator of w_N in the new basis by w^l_N):

$$w^{N+1}_{N+1}=M^{N+1}_{N+1}+\sum_{k=2}^{N}(-NB_N)^{N+1-k}M^k_{N+1}$$

$$=(-1)^N(NB_N)^{N+1}. \tag{27}$$

Note that we only use M^k_{N+1} for $k=2,\dots,N$ in the redefinition, since M^0_{N+1} and M^1_{N+1} are field-independent constants (9), (10), which are not generators of the w_{N+1}-algebra.

To obtain w_{N+1}^{N} we start with

$$M_{N+1}^{N}=M_{N}^{N}+\sum_{l=0}^{N-1}a_{N,l}{}^{N+1}M_{N}^{l}(B_{N})^{N-l}. \tag{28}$$

We then rewrite M_{N}^{N} using our result (25) with $N+1\to N$. In the second term of (28) we substitute (24). The result is

$$M_{N+1}^{N}=-\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}M_{N+1}^{k}[-(N-1)B_{N-1}]^{N-l}(B_{N})^{l-k}f_{l,k}{}^{N+1}$$
$$+\sum_{k=0}^{N-1}M_{N+1}^{k}(B_{N})^{N-k}\sum_{l=0}^{N-1}a_{N,l}{}^{N+1}f_{l,k}{}^{N+1}. \tag{29}$$

The last sum can be rewritten using

$$\sum_{l=0}^{N-1}a_{N,l}{}^{N+1}f_{l,k}{}^{N+1}=\delta_{Nk}-a_{N,N}{}^{N+1}f_{N,k}{}^{N+1}$$
$$=\delta_{Nk}-f_{N,k}{}^{N+1}. \tag{30}$$

Substituting this back in (29) gives finally

$$M_{N+1}^{N}=-\sum_{k=0}^{N-1}\sum_{l=0}^{N}M_{N+1}^{k}[-(N-1)B_{N-1}]^{N-l}(B_{N})^{l-k}f_{l,k}{}^{N+1}. \tag{31}$$

Again we can redefine to obtain the generator w_{N+1}^{N}:

$$w_{N+1}^{N}=-\sum_{l=0}^{N}[-(N-1)B_{N-1}]^{N-l}(B_{N})^{l}f_{l,0}{}^{N+1}. \tag{32}$$

Note that the $l=0$ term, which is independent of B_N, is equal to w_{N}^{N}.

This procedure can be continued for all spins. To continue to lower spins one needs to determine for each l the analogue of (25), (31), since, as for $l=N$, one uses the result for spin $l+1$ in the calculation for spin l. The redefinition then amounts to throwing away all contributions of M_{N+1}^{l} in the result except that of $l=0$. For spin $l=N-1$ we obtain in this way

$$w_{N+1}^{N-1}=-\sum_{k,l=0}^{N-1}[-(N-2)B_{N-2}]^{N-1-l}(B_{N-1})^{l-k}(B_{N})^{k}f_{l,k}{}^{N}f_{k,0}{}^{N+1}, \tag{33}$$

TABLE 3
Generators of w_N in our new basis for some low values of N

N	Generators of w_N
2	$w_2^2 = -(B_1)^2$
3	$w_3^2 = w_2^2 - 3(B_2)^2$ $w_3^3 = 8(B_2)^3$
4	$w_4^2 = w_3^2 - 6(B_3)^2$ $w_4^3 = w_3^3 + 12B_2(B_3)^2 + 20(B_3)^3$ $w_4^4 = -81(B_3)^4$
5	$w_5^2 = w_4^2 - 10(B_4)^2$ $w_5^3 = w_4^3 + 20B_2(B_4)^2 + 20B_3(B_4)^2 + 40(B_4)^3$ $w_5^4 = w_4^4 - 90(B_3)^2(B_4)^2 - 120B_3(B_4)^3 - 205(B_4)^4$ $w_5^5 = 1024(B_4)^5$

from which one can generalize to arbitrary spins:

$$w_{N+1}^{N-l} = -\sum_{k_1,\dots,k_{l+1}=0}^{N-l} \left[-(N-l-1)B_{N-l-1}\right]^{N-l-k_1}$$
$$\times (B_{N-l})^{k_1-k_2} \dots (B_{N-1})^{k_l-k_{l+1}} (B_N)^{k_{l+1}}$$
$$\times f_{k_1,k_2}{}^{N-l+1} f_{k_2,k_3}{}^{N-l+2} \dots f_{k_l,k_{l+1}}{}^{N} f_{k_{l+1},0}{}^{N+1}, \tag{34}$$

for $l = 0,\dots, N-2$. The highest spin generator w_{N+1}^{N+1} is given in (27). Again, if we select the term with vanishing power of B_N, we obtain w_N^{N-l}. It is a simple exercise to show that for $l = N-2$ the generator w_N^2 is equal to (11), i.e., the energy–momentum tensor is not modified by our redefinitions.

So in our new basis we have obtained in (27), (34) closed formulae for all generators of the classical w_N-algebra. Closure is guaranteed because of the closure of the algebra in the Miura basis. Of course, it is a formidable exercise to obtain the structure constants and the corresponding classical BRST charge explicitly for the complete w_N-algebra. In the next section, where we will address these problems, we will therefore limit ourselves to the v_{N+1}^N-algebra, which consists of the generators w_{N+1}^N and w_{N+1}^{N+1}.

For future reference we give explicit results for the redefined generators for the algebras w_N, $N = 2, 3, 4, 5$ in table 3.

3. The v_{N+1}^{N+1} and v_{N+1}^N subalgebras

The advantage of the new basis introduced in the previous section is that for each subalgebra of w_N one can define a nilpotent BRST charge Q. Clearly the

Q's, as the subalgebras, form a nested structure, in which Q_N^s, the BRST charge for the v_N^s-subalgebra, contains as contributions all $Q_N^{s'}$ for $s' \geqslant s$. Since each of these Q's is separately nilpotent, this nested structure should simplify the construction of, e.g., the physical states of the corresponding quantum theory, assuming of course that a quantum extension of this nested structure can be given. In this and the next section we will further discuss the classical structure of the algebra and its BRST current. The quantum extension will be considered in some specific examples in sect. 5.

For simplicity, let us start with the v_{N+1}^{N+1}-algebra. Its only generator is given in (27). It is a simple matter to calculate the Poisson bracket with itself. The basic OPE is given by *

$$B_m(z) B_n(w) \sim \frac{\delta_{mn}}{2n(n+1)} \frac{1}{(z-w)^2}. \tag{35}$$

For the generator w_{N+1}^{N+1} we then find

$$w_{N+1}^{N+1}(z) w_{N+1}^{N+1}(w) \sim \tfrac{1}{2}(-1)^N N^N (N+1) \times\left(\frac{1}{(z-w)^2} + \frac{1}{2}\frac{\partial}{z-w}\right)\left[(B_N)^{N-1} w_{N+1}^{N+1}(w)\right]. \tag{36}$$

The BRST current for the algebra (36) is easily obtained. Introducing the ghost and antighost pair (c_{N+1}, b_{N+1}), with the contraction

$$c_l(z) b_k(w) \sim \frac{\delta_{lk}}{z-w} \tag{37}$$

for any l, k, we obtain

$$j_{N+1}^{N+1} = c_{N+1} w_{N+1}^{N+1} - \tfrac{1}{4}(-1)^N N^N (N+1)(B_N)^{N-1} \partial c_{N+1}\, c_{N+1} b_{N+1}. \tag{38}$$

The pole of order one in the OPE of j with itself is a total derivative:

$$j(z) j(w) \sim \ldots + \frac{\partial(\ldots)}{z-w} + \ldots, \tag{39}$$

so that $Q = \oint \mathrm{d}z\, j(z)$ satisfies $\{Q, Q\} = 0$.

* In order to facilitate the transition to the quantum case, it is convenient to represent the Poisson brackets by Operator Product Expansions, in which only *single* contractions of fields are considered. After quantization *multiple* contractions have to be taken into account as well.

Thus we see that the BRST current for the v_{N+1}^{N+1}-algebra contains terms that are no more than cubic in the ghosts. This feature is no longer present when we consider the v_{N+1}^{s}-algebra.

For general N we will only consider the algebra containing the two generators (27) and (32). In this case we can obtain the structure constants of the algebra explicitly in terms of the coefficients f as given in (22). The v_{N+1}^{N}-algebra is given by the OPEs (36) and the following ones:

$$w_{N+1}^{N+1}(z)w_{N+1}^{N}(w) \sim -\tfrac{1}{2}\sum_{k=1}^{N}(-1)^{N-k}kf_{k,0}{}^{N+1}[(N-1)B_{N-1}(w)]^{N-k}$$
$$\times[B_N(w)]^{k-2}\left(\frac{w_{N+1}^{N+1}(w)}{N(z-w)^2}+\frac{\partial w_{N+1}^{N+1}(w)}{(N+1)(z-w)}\right), \quad (40)$$

$$w_{N+1}^{N}(z)w_{N+1}^{N}(w) \sim \left(\frac{1}{(z-w)^2}+\frac{1}{2}\frac{\partial}{z-w}\right)$$
$$\times\left(\tfrac{1}{2}\sum_{k=0}^{N-2}(-1)^{N+k+1}(N-k)(N-k-1)f_{k,0}{}^{N+1}\right.$$
$$\times[(N-1)B_{N-1}(w)]^{N-k-2}[B_N(w)]^k w_{N+1}^{N}(w)$$
$$+\frac{1}{2N}\sum_{k=0}^{N-3}(-1)^{N-k}(k+2)(N-2-k)f_{k+2,0}{}^{N+1}$$
$$\left.\times[(N-1)B_{N-1}(w)]^{N-k-3}[B_N(w)]^k w_{N+1}^{N+1}(w)\right). \quad (41)$$

Since the above algebra has been obtained from the Miura basis by a redefinition, closure is guaranteed. Nevertheless, it is interesting to check how restrictive the requirements of closure are on the coefficients in (36), (40) and (41). It is clear that in (36) there are no restrictions at all: for a single scalar we can always form an algebra with a single generator, for any spin. In (40) the sums must be such that negative powers of B_N are avoided. This is indeed the case, since $f_{1,0}=0$. One can easily check that this is the only condition on the coefficients f required for closure. In (41) the situation is more complicated. One can parametrize the right hand side of (41) with an expansion in powers of B_{N-1} and B_N with arbitrary coefficients, multiplying the generators w_{N+1}^{N} and w_{N+1}^{N+1}. It turns out that the requirements of closure can be solved for all unknown coefficients, but that two consistency equations remain. In terms of the coefficients f these are two quadratic identities of the form

$$\frac{(N-1)}{N}\sum_{k=0}^{N-1}(k+1)(N-k)f_{k,0}{}^{N+1}f_{N-1-k,0}{}^{N+1}$$
$$+\frac{1}{N(N+1)}\sum_{k=0}^{N-1}(k+1)(N-k)f_{N-k,0}{}^{N+1}f_{k+1,0}{}^{N+1}$$
$$-\sum_{k=0}^{N-2}(N-k)(N-k-1)f_{k,0}{}^{N+1}f_{N-k-1,0}{}^{N+1}=0, \tag{42}$$

$$\frac{(N-1)}{N}\sum_{k=0}^{N}k(N-k)f_{k,0}{}^{N+1}f_{N-k,0}{}^{N+1}$$
$$+\frac{1}{N(N+1)}\sum_{k=0}^{N-2}(k+2)(N-k)f_{k+2,0}{}^{N+1}f_{N-k,0}{}^{N+1}$$
$$+\sum_{k=0}^{N-2}(N-k)(N-k-1)f_{k,0}{}^{N+1}f_{N-k,0}{}^{N+1}=0. \tag{43}$$

Calculations for the coefficients $f_{k,0}{}^{N+1}$ for general N and $k=0,1,\ldots,$ are done using the explicit form (22). In some calculations, such as in the check of (42), (43) we also need for general N the coefficients $f_{N-k,0}{}^{N+1}$ for $k=0,1\ldots.$ We have then used the following representation of the f's:

$$f_{N-k,0}{}^{N+1}=\sum_{l=0}^{k}\binom{N-l}{k-l}\frac{(N)^k}{(1+N)^{k+1}}$$
$$\times\left[1-(-N)^{N-k+1}\sum_{j=0}^{l}\binom{N-k+1}{j}\left(-\frac{1+N}{N}\right)^j\right]. \tag{44}$$

The BRST charge for the v_{N+1}^{N}-algebra is much more complicated than (38) for the v_{N+1}^{N+1}-algebra. In particular, there will be ghost contributions of higher order than cubic terms. The same applies to the BRST charge for the general v_{N+1}^{l}-algebra. We have not attempted to obtain the BRST current j_{N+1}^{l} for general l and N. Instead, we will give in the next section explicit expressions for some specific values of l and N.

Using a dimensional argument, it is possible to give a limit on the terms of higher order in the ghost fields that may appear in the BRST charge. Let us briefly present this argument for the v_{N+1}^{N}-algebra. There we have two pairs of ghosts, (b_{N+1}, c_{N+1}) and (b_N, c_N). The conformal spin of the BRST current equals one, the ghost fields b_n and c_n have spins n and $1-n$. Also, Q has ghost number one. A $(2n+1)$-order ghost contribution to Q for the v_{N+1}^{N}-algebra would be of the form

$$(b_{N+1})^k (b_N)^l (c_{N+1})^p (c_N)^q, \quad k+l=n, \quad p+q=n+1, \tag{45}$$

where the powers of the anticommuting ghost fields are given by, e.g., $(b_n)^k \equiv b_n(\partial b_n)\dots(\partial^{k-1} b_n)$. The conformal weight s_b and s_c of the product of all b- and c-ghosts in (45) is then

$$s_b = k^2 + k(1-n) + nN + \tfrac{1}{2}n(n-1),$$

$$s_c = p^2 - 2p - np - (n+1)N + \tfrac{1}{2}(n+1)(n+2).$$

The minimum values for s_b and s_c are reached for $k = \frac{1}{2}(n-1)$ and $p = \frac{1}{2}(n+2)$, respectively. The value of the sum of the minima of s_b and s_c is given by

$$s_{\min} = \tfrac{1}{4}(2n^2 + 2n - 1) - N. \tag{46}$$

For such a ghost term to exist in the BRST current we must have $s_{\min} \leqslant 1$, so that it is possible to obtain $s_Q = 1$. Therefore we should have

$$2n^2 + 2n - 1 \leqslant 4(N+1) \tag{47}$$

for the v_{N+1}^{N}-algebra. For the v_4^3-algebra this implies that terms of fifth order in the ghosts can be written down. However, as we shall see in the next section, only cubic ghost terms actually appear. For the v_5^4-algebra fifth-order ghost terms are possible, but seventh-order ghost terms are not. In that case we find that the fifth-order terms are required in the BRST charge.

Clearly, the dimensional argument can be extended to v_{N+1}^{l}-algebras.

4. Classical BRST charges

In this section we give explicit expressions for the BRST charges of w_3, w_4 and the subalgebra $v_5^4 \subset w_5$ in the new basis, thereby making explicit the nested structure (3). To obtain there BRST charges, it is convenient to use an iterative procedure. Starting from the terms in the BRST charge that are linear in the ghosts (the terms containing the generators), one obtains higher-order ghost terms by demanding nilpotency. In the next order one finds that the coefficients multiplying the cubic ghost terms are the structure constants of the algebra (in the new basis). Since we are dealing with field-dependent structure constants, it may be necessary to add higher-order ghost terms as well.

For pedagogical reasons, we will discuss first the case of the w_3-algebra in somewhat more detail [12]. The generators of the w_3-algebra are given in table 3. Using (6), these generators can be written as

$$T = -2w_3^2 = -\tfrac{1}{2}(\partial\phi_1)^2 - \tfrac{1}{2}(\partial\phi_2)^2,$$

$$W = -2\sqrt{3}\; w_3^3 = \tfrac{2}{3}i(\partial\phi_2)^3. \tag{48}$$

Note that the generators w_3^2 and w_3^3 have been rescaled. This makes T an energy–momentum tensor generating the Virasoro algebra. For W the rescaling is just a matter of convenience.

The OPE of W with itself is *

$$W(z)W(w) \sim \left(\frac{1}{(z-w)^2} + \frac{1}{2}\frac{\partial}{z-w}\right)(-6i\partial\phi_2\, W). \tag{49}$$

From this algebra one can read off the BRST current $j_3(z)$ up to third-order ghost terms, and it turns out that no higher-order terms are needed. It can be written as $j_3(z) = j_3^2(z)$, with

$$j_3^3(z) = c_3 W - 3i\partial\phi_2\; c_3\partial c_3\; b_3,$$

$$j_3^2(z) = c_2\left(T + T_{c_3,b_3} + \tfrac{1}{2}T_{c_2,b_2}\right) + j_3^3(z), \tag{50}$$

where we defined the ghost energy–momentum tensors

$$T_{c_s,b_s} = -s b_s\partial c_s - (s-1)\partial b_s\; c_s \tag{51}$$

for arbitrary spin s. The expression for $j_3^3(z)$ agrees with the formula for general N given in (38). Note that the two charges Q_3^3 and $Q_3^2 - Q_3^3$ are separately nilpotent.

It is instructive to compare the above result for the BRST charge in the new basis with the one in the Miura basis. The two expressions are related to each other by a canonical transformation in the extended phase space [23]. It turns out

* The OPEs involving the energy–momentum tensor T are standard and not given here.

that the canonical transformation that relates (50) to the Miura basis is generated by

$$G = i\partial\phi_2\ c_3 b_2. \tag{52}$$

The exponential action of the generator G on an extended phase space function F is, in OPE language,

$$F(w) \to F(w) + \oint \frac{\mathrm{d}z}{2\pi i} G(z) F(w) + \frac{1}{2!} \oint \frac{\mathrm{d}z}{2\pi i} G(z) \oint \frac{\mathrm{d}x}{2\pi i} G(x) F(w) + \dots. \tag{53}$$

This results in the following transformations of the basis fields [12]:

$$\begin{aligned}
\tilde{c}_2 &= c_2 + i\partial\phi_2\ c_3 + \tfrac{1}{2} c_3 \partial c_3\ b_2, \\
\tilde{b}_2 &= b_2, \\
\tilde{c}_3 &= c_3, \\
\tilde{b}_3 &= b_3 - i\partial\phi_2\ b_2 + \tfrac{1}{2} b_2 \partial b_2\ c_3, \\
\partial\tilde{\phi}_2 &= \partial\phi_2 + i\partial(b_2 c_3),
\end{aligned} \tag{54}$$

where the tilde indicates the fields in the Miura basis. Due to anticommutativity of the ghost variables, only the first few terms in (53) contribute to (54). The BRST current (50) now transforms into its Miura form (suppressing the tilde on both fields and generators) [7]

$$j(z) = c_2\left(T + T_{c_3,b_3} + \tfrac{1}{2} T_{c_2,b_2}\right) + c_3 W + \tfrac{1}{2} c_3 \partial c_3\ b_2 T. \tag{55}$$

Note that the nested structure is absent in the Miura basis: the BRST current (55) cannot be written as the sum of two separate nilpotent currents.

The advantage of using the new basis instead of the Miura basis is even more apparent when we discuss w_4. The BRST charge for w_4 in the Miura basis has recently been calculated in refs. [10,11]. The authors of refs. [10,11] find that the BRST current contains terms up to seventh order in the ghosts. As we will show below, in the new basis we not only make the nested structure of the BRST charge explicit, but we furthermore find that in the new basis all higher-order ghost terms vanish and that at most trilinear ghost terms occur.

The generators of the w_4-algebra in the new basis are given in table 3:

$$
\begin{aligned}
T &= -2w_4^2 = -\tfrac{1}{2}(\partial\phi_1)^2 - \tfrac{1}{2}(\partial\phi_2)^2 - \tfrac{1}{2}(\partial\phi_3)^2, \\
W &= 3i\sqrt{3}\ w_4^3 = (\partial\phi_2)^3 + \tfrac{3}{4}\partial\phi_2(\partial\phi_3)^2 + \tfrac{5}{8}\sqrt{2}\,(\partial\phi_3)^3, \\
V &= -\tfrac{64}{9}w_4^4 = (\partial\phi_3)^4,
\end{aligned}
\tag{56}
$$

where we have made some convenient rescalings of the generators. The nontrivial OPEs (not involving the energy–momentum tensor) among these generators are

$$
\begin{aligned}
W(z)W(w) &\sim \left(\frac{1}{(z-w)^2} + \frac{1}{2}\frac{\partial}{z-w}\right)\left(-9\partial\phi_2\, W - \tfrac{243}{32}V\right), \\
W(z)V(w) &\sim \frac{-6\partial\phi_2\, V - 9\partial\phi_3\, V}{(z-w)^2} \\
&\quad + \frac{-\tfrac{3}{2}\partial\phi_2\, \partial V - 6\partial^2\phi_2\, V - \tfrac{18}{5}\partial(\partial\phi_3\, V)}{(z-w)}, \\
V(z)V(w) &\sim \left(\frac{1}{(z-w)^2} + \frac{1}{2}\frac{\partial}{z-w}\right)\left[-16(\partial\phi_3)^2 V\right].
\end{aligned}
\tag{57}
$$

From this algebra, one can read off the BRST current $j_4(z)$ up to third-order ghost terms, and it turns out that no higher-order terms are needed. It can be written as $j_4(z) = j_4^2(z)$, with

$$
\begin{aligned}
j_4^4(z) &= c_4 V - 8(\partial\phi_4)^2 c_4\partial c_4\, b_4, \\
j_4^3(z) &= c_3 W - \tfrac{9}{2}\partial\phi_2\, c_3\partial c_3\, b_3 - \tfrac{243}{64}c_3\partial c_3\, b_4 - \tfrac{9}{2}\partial\phi_2\, c_3 c_4\partial b_4 \\
&\quad - 6\partial\phi_2\, c_3\partial c_4\, b_4 + \tfrac{9}{2}\sqrt{2}\,\partial\phi_3\, \partial c_3\, c_4 b_4 - 3\sqrt{2}\ \partial\phi_3\, c_3 c_4\partial b_4 \\
&\quad + j_4^4(z), \\
j_4^2(z) &= c_2\left(T + T_{c_3,b_3} + T_{c_4,b_4} + \tfrac{1}{2}T_{c_2,b_2}\right) + j_4^3(z).
\end{aligned}
\tag{58}
$$

The nested structure of the BRST charges manifests itself through the fact that the BRST charges associated with j_4^4, j_4^3 and j_4^2 are all nilpotent. Furthermore, $j_4^2 - j_4^3$ is the BRST current of the Virasoro algebra, and is separately nilpotent.

So far, for w_3 and w_4 in the new basis, we have not encountered terms in the BRST current that are of higher than third order in the ghosts. This is not always the case. The most simple example where one has to go beyond the trilinear ghost terms, even in the new basis, is given by the BRST current corresponding to the v_5^4 subalgebra of w_5. In particular, we find that the BRST current j_5^4 contains terms quintic in the ghosts. The nested structure of the currents is given by

$$
\begin{aligned}
j_5^5 &= c_5\left[\tfrac{4}{125} i\sqrt{10}\,(\partial\phi_4)^5\right] + \tfrac{2}{5} i\sqrt{10}\; c_5' c_5 b_5 (\partial\phi_4)^3 , \\
j_5^4 &= c_4\left[-\tfrac{9}{64}(\partial\phi_3)^4 - \tfrac{3}{32}(\partial\phi_3)^2(\partial\phi_4)^2 - \tfrac{1}{40}\sqrt{15}\;\partial\phi_3(\partial\phi_4)^3 - \tfrac{41}{320}(\partial\phi_4)^4\right] \\
&\quad + \left[-\tfrac{9}{8}(\partial\phi_3)^2 - \tfrac{1}{8}(\partial\phi_4)^2\right] c_4' c_4 b_4 \\
&\quad + \left[-\tfrac{3}{8} i\sqrt{10}\;\partial\phi_4 - \tfrac{5}{8} i\sqrt{6}\;\partial\phi_3\right] c_4' c_4 b_5 \\
&\quad + \left[-\tfrac{15}{16}(\partial\phi_3)^2 - \tfrac{1}{8}\sqrt{15}\;\partial\phi_3\,\partial\phi_4 - \tfrac{19}{16}(\partial\phi_4)^2\right] c_5' c_4 b_5 \\
&\quad + \left[\tfrac{1}{4}\sqrt{15}\;\partial\phi_3\,\partial\phi_4 + \tfrac{11}{8}(\partial\phi_4)^2\right] c_5 c_4' b_5 \\
&\quad + \left[-\tfrac{3}{4}(\partial\phi_3)^2 - \tfrac{1}{8}(\partial\phi_4)^2\right] c_5 c_4 b_5' \\
&\quad + \tfrac{1}{4}\sqrt{15}\;\partial^2\phi_3\,\partial\phi_4\; c_5 c_4 b_5 \\
&\quad - \tfrac{5}{4} c_5' c_5 c_4 b_5' b_5 + \tfrac{5}{4} c_5' c_4' c_4 b_5 b_4 + c_5 c_4' c_4 b_5' b_4 \\
&\quad + j_5^5 .
\end{aligned}
\tag{59}
$$

5. Quantization

So far, our discussion has basically been at the classical level. In this section we will discuss some aspects of the quantisation, in particular the construction of the quantum BRST operators. The results of this section indicate that the nested structure found at the classical level survives the quantization.

Our strategy is to use the classical results of the previous sections as a starting point for the construction of the quantum BRST operators *. In practice, the easiest way to obtain explicit expressions for the BRST operators for low values of N is to parametrize all possible quantum corrections to the classical BRST charge, and then to determine the coefficients occurring in the ansatz by requiring nilpotency of the quantum BRST operator. We will use this explicit method to discuss the quantization of the w_4-algebra **.

* Note that we write the quantum expressions with boldface.

** The quantization of the w_3-algebra in the new basis was done in ref. [12] and we will not repeat it here.

We would like to stress that the use of the new basis greatly facilitates the construction of the quantum BRST operator. The nested structure enables one to construct the BRST operator in an iterative way. One starts with $\mathbf{Q}_N^N$, the BRST operator corresponding to the highest spin generator of W_N. This will depend on only one scalar, and on the spin-N ghosts b_N, c_N. Next one goes on to $\mathbf{Q}_N^{N-1}$, which will depend on one additional scalar and the spin-$(N-1)$ ghost pair as well. In this way, one obtains at each level a nilpotent BRST operator, which contains the operators of the higher spin subalgebras. In the last step one obtains the BRST operator of the complete W_N-algebra.

For $N=4$ the quantum extension $\mathbf{j}_4^4$ of the highest spin contribution j_4^4 to the classical BRST current was already given in ref. [18]. We now give the result for the full w_4-algebra, including also $\mathbf{j}_4^3$ and $\mathbf{j}_4^2$:

$$\begin{aligned}\mathbf{j}_4^4 = {}& c_4\Big[(\partial\phi_3)^4 + \tfrac{18}{5}\sqrt{15}\,\partial^2\phi_3(\partial\phi_3)^2 + \tfrac{41}{5}\partial^2\phi_3\,\partial^2\phi_3 \\ & + \tfrac{124}{15}\partial^3\phi_3\,\partial\phi_3 + \tfrac{23}{75}\sqrt{15}\,\partial^4\phi_3\Big] \\ & - 8(\partial\phi_3)^2 c_4c_4'b_4 + \tfrac{8}{5}\sqrt{15}\,\partial^2\phi_3 c_4c_4'b_4 \\ & + \tfrac{16}{5}\sqrt{15}\,\partial\phi_3\,c_4c_4''b_4 + \tfrac{4}{5}c_4c_4'''b_4 - \tfrac{16}{3}c_4c_4'b_4'',\end{aligned} \tag{60}$$

$$\begin{aligned}\mathbf{j}_4^3 = {}& c_3\Big[(\partial\phi_2)^3 + \tfrac{3}{4}\partial\phi_2(\partial\phi_3)^2 + \tfrac{5}{8}\sqrt{2}\,(\partial\phi_3)^3 \\ & + \tfrac{27}{20}\sqrt{30}\,\partial\phi_2\,\partial^2\phi_2 + \tfrac{27}{20}\sqrt{15}\,\partial\phi_2\,\partial^2\phi_3 + \tfrac{81}{40}\sqrt{30}\,\partial\phi_3\,\partial^2\phi_3 \\ & + \tfrac{93}{40}\partial^3\phi_2 + \tfrac{69}{10}\sqrt{2}\,\partial^3\phi_3\Big] \\ & - \tfrac{9}{2}\partial\phi_2\,c_3c_3'b_3 - \tfrac{27}{40}\sqrt{30}\,c_3''c_3b_3 - \tfrac{243}{64}c_3c_3'b_4 \\ & - \tfrac{9}{2}\partial\phi_2\,c_3c_4b_4' - 6\partial\phi_2\,c_3c_4'b_4 - \tfrac{81}{40}\sqrt{30}\,c_3''c_4b_4 \\ & + \tfrac{27}{40}\sqrt{30}\,c_3c_4''b_4 + \tfrac{9}{2}\sqrt{2}\,\partial\phi_3\,c_3'c_4b_4 - 3\sqrt{2}\,\partial\phi_3\,c_3c_4'b_4 \\ & + \mathbf{j}_4^4,\end{aligned} \tag{61}$$

$$\begin{aligned}\mathbf{j}_4^2 = {}& c_2\Big[-\tfrac{1}{2}(\partial\phi_1)^2 - \tfrac{1}{2}(\partial\phi_2)^2 - \tfrac{1}{2}(\partial\phi_3)^2 \\ & \pm\tfrac{9}{20}\sqrt{10}\,\partial^2\phi_1 - \tfrac{9}{20}\sqrt{30}\,\partial^2\phi_2 - \tfrac{9}{10}\sqrt{15}\,\partial^2\phi_3\Big] \\ & + c_2c_2'b_2 + 3c_2c_3'b_3 + 2c_2c_3b_3' + 4c_2c_4'b_4 + 3c_2c_4b_4' \\ & + \mathbf{j}_4^3.\end{aligned} \tag{62}$$

It turns out that there exists another nilpotent BRST charge for the quantum W_4-algebra which has a different sign for the background charge of ϕ_2. So $\mathbf{j}_4^2 - \mathbf{j}_4^3$ is the same except that

$$-\tfrac{9}{20}\sqrt{30}\ \partial^2\phi_2 \to +\tfrac{9}{20}\sqrt{30}\ \partial^2\phi_2. \tag{63}$$

Using the other choice of sign for the background charge, we find that $\mathbf{j}_4^4$ is the same but that $\mathbf{j}_4^3$ is now given by

$$\begin{aligned}
\mathbf{j}_4^3 = {} & c_3\Big[(\partial\phi_2)^3 + \tfrac{3}{4}\partial\phi_2(\partial\phi_3)^2 + \tfrac{5}{8}\sqrt{2}\,(\partial\phi_3)^3 \\
& - \tfrac{27}{20}\sqrt{30}\ \partial\phi_2\ \partial^2\phi_2 + \tfrac{27}{20}\sqrt{15}\ \partial\phi_2\ \partial^2\phi_3 + \tfrac{27}{20}\sqrt{30}\ \partial\phi_3\ \partial^2\phi_3 \\
& + \tfrac{93}{40}\partial^3\phi_2 - \tfrac{177}{80}\sqrt{2}\ \partial^3\phi_3\Big] \\
& - \tfrac{9}{2}\partial\phi_2\ c_3c_3'b_3 + \tfrac{27}{40}\sqrt{30}\ c_3''c_3b_3 - \tfrac{243}{64}c_3c_3'b_4 \\
& - \tfrac{9}{2}\partial\phi_2\ c_3c_4b_4' - 6\partial\phi_2\ c_3c_4'b_4 + \tfrac{27}{40}\sqrt{30}\ c_3c_4''b_4 \\
& - \tfrac{27}{40}\sqrt{30}\ c_3'c_4'b_4 + \tfrac{9}{2}\sqrt{2}\ \partial\phi_3\ c_3'c_4b_4 - 3\sqrt{2}\ \partial\phi_3\ c_3c_4'b_4 \\
& + \mathbf{j}_4^4.
\end{aligned} \tag{64}$$

It is not clear to us whether this second solution can be related to the first one by a canonical transformation.

Our result for the W_4-algebra is based on one of the solutions for $\mathbf{j}_4^4$ obtained in ref. [18], namely the solution where the background charge of the fields are the same as in the Miura basis. Besides this solution, the authors of ref. [18] found one additional solution for $\mathbf{j}_4^4$ with a different value of the background charge for ϕ_3. We have attempted to extend also this solution with a $\mathbf{j}_4^3$ and $\mathbf{j}_4^2$. However, the calculation shows that for this additional solution such an extension is impossible.

The result (60) for $\mathbf{j}_4^4$ provides a nice example of a phenomenon which we discussed in the introduction, namely that at the quantum level consistency of the theory requires the existence of a nilpotent BRST operator but not of a closed quantum algebra. Indeed, although a nilpotent BRST operator $\mathbf{Q}_4^4$ exists, it is not possible to find a quantum extension of the classical v_4^4-subalgebra in the full Hilbert space *.

The quantum BRST operator for the W_4-algebra in the Miura basis has recently been obtained in refs. [10,11]. Due to the complexity of their result it is hard to

* Note that it may be possible to obtain closure by introducing additional generators besides the spin-four generator in the quantum algebra. This has been done for W_3 in ref. [20].

compare with our $N=4$ BRST current (60)–(62) but we expect that the two expressions are related through a canonical transformation.

6. W-strings and minimal models

As we already discussed in the introduction it has become more and more clear that there exists a relation between the spectra of W-strings and certain minimal models [15,16,12,17–19,13,20]. In this section we will suggest a very general relationship between W-strings and minimal models by exploiting the nested structure discussed in this paper. It would be interesting to see whether our suggestions can be confirmed by explicit calculations of the spectra of W-strings. We will first discuss the case of critical W-strings and then investigate non-critical W-strings.

6.1. CRITICAL W-STRINGS

By a "critical" W-string, we mean that we work with only one copy of a W-algebra. This W-algebra is realized in terms of so-called "matter" fields, the "Liouville" fields being absent *.

As a warming-up exercise we first consider the BRST operator $\mathbf{Q}_N^N$, corresponding to the highest spin of the W_N-algebra. This operator has already been constructed for $N \leqslant 6$ in ref. [18]. The result, for general N, is that $\mathbf{Q}_N^N$ depends on a single scalar field ϕ_{N-1}, and on the ghost fields b_N, c_N of the spin-N symmetries. It is nilpotent, and commutes with an energy–momentum tensor depending on the same fields, of the form

$$\mathbf{T}_N^N = -\tfrac{1}{2}(\partial\phi_{N-1})^2 - \alpha_{N-1}\partial^2\phi_{N-1} - Nb_N\partial c_N - (N-1)(\partial b_N)c_N. \qquad (65)$$

$[\mathbf{Q}_N^N, \mathbf{T}_N^N] = 0$ determines the background charge α_{N-1}. For general N

$$(\alpha_{N-1})^2 = \frac{(N-1)(2N+1)^2}{4(N+1)} \qquad (66)$$

should be one of the allowed values of the background charge. This has been verified for $N \leqslant 6$ in ref. [18]. The authors of ref. [18] find that also other values of

* The distinction between "matter" and "Liouville" fields is a little ambiguous, since in the case of W-algebras, some of the "matter" fields *must* have a background charge and might therefore also be called "Liouville" fields. We will adopt a convention where the "Liouville" fields are introduced later as a separate realization of the W-algebra (see below). This definition of a "non-critical" W-string is in accordance with the one used in ref. [9].

the background charge are possible. With the value of α_{N-1} as in (66) we find that the total central charge of $\mathbf{T}_N^N$ is *

$$c_N^N \equiv 1 + 12(\alpha_{N-1})^2 - 2(6N^2 - 6N + 1) = \frac{2(N-2)}{N+1}. \tag{67}$$

This value corresponds to the central charge of a minimal model of the W_{N-1}-algebra. In general, the unitary minimal models of the W_M-algebra are characterized by central charges (for any integer $q > M$)

$$c_{M,q} \equiv (M-1)\left(1 - \frac{M(M+1)}{q(q+1)}\right), \tag{68}$$

so that (67) corresponds to $c_{N-1,N}$. For $N = 3$, (67) then corresponds to the central charge of a Virasoro minimal model, namely the $c = \frac{1}{2}$ Ising model. Mounting evidence that the cohomology of $\mathbf{Q}_3^3$ indeed produces the result of the $c = \frac{1}{2}$ Ising model has been given in refs. [12,17–19,20]. The relationship between critical W_N-strings and minimal models for general N was further explored in refs. [22,24,20]. In particular, it was noted that in a particular realization of W_N [22], the scalar fields $\phi_2, \dots, \phi_{N-1}$, together with the ghost fields corresponding to the spins $3, \dots, N$, form an energy–momentum tensor with central charge

$$c_N^3 = 1 - \frac{6}{N(N+1)}, \tag{69}$$

corresponding to the $q = N$ minimal model of the Virasoro algebra. We will now show, using the nested structure of the W_N-algebra, that it is possible to interpolate between c_N^N and c_N^3.

The background charges of the $N - 1$ scalar fields that realize the W_N-algebras are known in the Miura basis [15,22]. The iterative relation which determines the matter part of the energy–momentum tensor is in the quantum case

$$\mathbf{T}_N = \mathbf{T}_{N-1} - \tfrac{1}{2}(\partial\phi_{N-1})^2 + ix\sqrt{\tfrac{1}{2}(N-1)N}\; \partial^2\phi_{N-1}, \tag{70}$$

where x is a parameter. The total central charge of all scalars is then

$$c_{\mathrm{m}} = \sum_{n=1}^{N-1} \left[1 - 6x^2 n(n+1)\right] = (N-1)\left[1 - 2x^2 N(N+1)\right]. \tag{71}$$

* Note that this is exactly the value of the central charge corresponding to a SU$(N-1)$ parafermionic theory [19].

On the other hand, the total central charge of the ghost fields is given by

$$c_{\mathrm{gh}} = -2 \sum_{n=2}^{N} (6n^2 - 6n + 1) = -2(N-1)(2N^2 + 2N + 1). \tag{72}$$

Criticality therefore requires

$$x = i(2N+1)\sqrt{\frac{1}{2N(N+1)}}. \tag{73}$$

This determines the background charges of all scalar fields ϕ_n:

$$\alpha_n = \frac{2N+1}{2}\sqrt{\frac{n(n+1)}{N(N+1)}}. \tag{74}$$

This indeed gives (66) for $n = N-1$.

In sect. 2 we performed a redefinition of the generators of the classical w_N-algebra, starting from the classical form of the Miura basis. In this redefinition the energy–momentum tensor was not modified. We conjecture that similarly, the energy–momentum tensor in our nested basis will have the same form as in the quantum Miura basis *. The background charges of all scalar fields are then known, and we can analyze the central charge of that part of the total energy–momentum tensor that corresponds to the BRST operator $\mathbf{Q}_N^n$, and contains the matter fields $\phi_{n-1}, \dots, \phi_{N-1}$ and the ghost fields b_n, $c_n, \dots, b_N$, c_N. The total central charge is given by

$$\begin{aligned} c_N^n &= -2 \sum_{k=n}^{N} (6k^2 - 6k + 1) + \sum_{k=n-1}^{N-1} \left[1 + 12(\alpha_k)^2\right] \\ &= (n-2)\left(1 - \frac{n(n-1)}{N(N+1)}\right). \end{aligned} \tag{75}$$

This is equal to $c_{n-1,N}$, the central charge of the $q = N$ minimal model of the W_{n-1}-algebra. For $n = 2$ we find of course that $c_N^2 = 0$, because this case corresponds to the critical W_N-string. For $n = N$ we obtain (67). Note that the relation (75) between critical W_N-strings and minimal models of the W_{n-1}-algebra was suggested before, from a different point of view, in ref. [22].

* This assumption has been verified for $N = 3$ and $N = 4$ (see sect. 5) and for the highest spin generator for $N \leqslant 6$ [18]. Note that the discussion of the highest spin generator given in ref. [20] depends on the same assumption.

To summarize, the nested structure of the W_N-algebra and of the corresponding BRST operators clarifies the connection with minimal models.

6.2. NON-CRITICAL W-STRINGS

The situation is different for the so-called non-critical W_N-string [9]. In the case of the non-critical string we have classically two copies of a w_N-algebra, which we call w_{m} and w_ℓ, for matter and Liouville, respectively. Although the algebra is nonlinear, a combined algebra can nevertheless be formed with generators $w_N^k \equiv (w_{\mathrm{m}})_N^k + i^{k-2}(w_\ell)_N^k$. In the case $N=3$ the quantum BRST operator for this system was constructed in refs. [9,25]. The non-critical W_N-string is characterized by the central charges of the matter and Liouville sectors, c_{m} and c_ℓ respectively. To allow for a nilpotent BRST operator these central charges must satisfy (see (72))

$$c_{\mathrm{m}} + c_\ell = 2(N-1)(2N^2+2N+1). \tag{76}$$

We can again go to the nested basis discussed in previous sections, but the required redefinitions can only be made for either the matter or the Liouville sector. Let us choose the Liouville sector *. Then c_ℓ is given by (71)

$$c_\ell = (N-1)\left[1 - 2x^2 N(N+1)\right], \tag{77}$$

but, in contradistinction to the situation considered in sect. 5, (76) is now not sufficient to express x in terms of N. Therefore, the non-critical strings of ref. [9] have one arbitrary parameter, x, which makes it possible to avoid the relation with minimal models. If we choose our nested basis for the Liouville sector, then we can make a nilpotent BRST operator depending on the field ϕ_{N-1}, one of the Liouville scalars, the spin-N ghost and antighost fields and all fields of the matter sector. The total central charge corresponding to this case is

$$\begin{aligned} c_N^N &= c_{\mathrm{m}} + 1 - 6x^2N(N-1) - 2(6N^2-6N+1) \\ &= (N-2)\left[(2N-1)^2 + 2N(N-1)x^2\right], \end{aligned} \tag{78}$$

the analogue of (67). For general x this does not correspond to a minimal model.

By choosing x appropriately we can of course obtain a minimal model. In particular, we get the qth unitary minimal model of the W_{N-1}-string by choosing x

* The discussion below can be repeated for the case where a nested basis is chosen in the matter sector.

equal to *

$$x^2 = -2 - \frac{1}{2q(q+1)}. \tag{79}$$

Note that in this case c_{m}, which can be determined from (76), (77), is equal to

$$c_{\mathrm{m}} = (N-1)\left(1 - \frac{N(N+1)}{q(q+1)}\right), \tag{80}$$

which corresponds to the qth minimal model of the W_N-string. The values of x given in (79) were also considered in refs. [26,27], where the cohomology of the non-critical W_3-string was investigated.

Using the nested basis in the Liouville sector we get a series of nested BRST operators, $\mathbf{Q}_N^n$, depending on all matter fields, the scalars $\phi_{n-1}, \dots, \phi_{N-1}$ of the Liouville sector and the ghost and antighost fields of the spin-$n, \dots, N$ symmetries. For general x the central charge of the corresponding energy–momentum tensor is

$$c_N^n = (n-2)\left[(2n-1)^2 + 2n(n-1)x^2\right]. \tag{81}$$

When x is given by (79) this corresponds to the qth unitary minimal model of the W_{n-1}-algebra. This relation with minimal models extends the discussion in ref. [13].

Note that in the present case of the non-critical string we have the additional freedom of selecting the minimal model: the value of q is arbitrary in (79), while in (75) we necessarily obtained $q = N$. This is to be expected since for $q = N$ we have $c_{\mathrm{m}} = 0$ and the theory effectively reduces to the critical W-string. As mentioned in the previous footnote, for the non-critical W-string non-unitary minimal models can be considered in the same way.

We conclude that in the case of the non-critical string the relation with minimal models is not forced upon us, and that the non-critical string therefore allows for a much wider class of models than the critical string. With a particular choice of the parameter x we obtain results similar to those in the critical case.

It would be very interesting to investigate in further detail the relations between (critical and/or non-critical) W-strings and minimal models. The fact that in the non-critical case this relationship can be avoided should have some significance. Probably the best way to proceed is by investigating the cohomology of the different BRST operators in the "nested" basis discussed in this paper. An interesting simple example where the spectrum can be calculated is provided by

* Non-unitary minimal models can be obtained by choosing more generally $x^2 = -2 - \frac{1}{2}(Q_{\mathrm{M}})^2$, with $Q_{\mathrm{M}} = \sqrt{p/q} - \sqrt{q/p}$ [13]. For comparison with subsect. 6.1 we will limit ourselves in the text to unitary models ($p = q + 1$), but the results which follow can all be easily extended to the non-unitary case.

taking a non-critical W_3-string where the Liouville sector is realized by just one scalar [28].

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